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CONVERGENCE AND TRUNCATION ERROR
STUDY IN TIME SERIES EXPANSIONS 9

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INTRODUCTION

This study determines the radius of convergence of the Taylor series expansion in time of the coordinates representing the motion of an artificial satellite along an elliptic path.

An evaluation of the truncation errors of the time series is also given to $O(t^3)$ and $O(t^5)$, where t is the length of the time interval, for the case of the smallest radius of convergence.

The results of the investigation are shown in graphs 1 through 5. From these, the radius of convergence and the truncation errors may be deduced as a function of the eccentricity of the orbit, the semi-major axis and the position of the body in its path. Results for orbits around the Earth and lunar trajectories are presented separately.

Analytic Outline

The Cartesian coordinates of a point moving in a Keplerian elliptic orbit may be expressed in time-series expansions. This paper is an investigation of the convergence and truncation error in these expansions.

A. Radius of Convergence of Time-series.

1. The coordinates of elliptic motion at time t ,

$$\xi = r \cos \vartheta = a (\cos E - e) \quad (1)$$

$$\eta = r \sin \vartheta = a (1 - e^2)^{1/2} \sin E$$

may be expressed by:

$$\xi = f \xi_0 + g \dot{\xi}_0 \quad (2)$$

$$\eta = f \eta_0 + g \dot{\eta}_0$$

where (ξ_0, η_0) and $(\dot{\xi}_0, \dot{\eta}_0)$ are the position and velocity at time t_0 .

The functions f and g are the well-known Lagrangian power series in

$t - t_0$. Putting:

$$\tau = t - t_0 \quad (\text{in normalized units})$$

$$r_0^2 = \xi_0^2 + \eta_0^2 \quad (3)$$

$$V_0^2 = \dot{\xi}_0^2 + \dot{\eta}_0^2$$

$$\sigma_0 = \xi_0 \dot{\xi}_0 + \eta_0 \dot{\eta}_0$$

$$u_0 = \frac{1}{2} r_0^{-3}$$

$$p_0 = \sigma_0 r_0^{-2}$$

$$q_0 = V_0^2 r_0^{-2} - (2u_0 + p_0^2)$$

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it may be shown that:

$$f = 1 + \sum_{k=2}^{\infty} f_k \tau^k \quad (4)$$

$$g = \tau + \sum_{k=3}^{\infty} g_k \tau^k$$

where the first six coefficients of the power series are given by:

$$\begin{aligned} f_2 &= -u_0 \\ f_3 &= u_0 p_0 \\ f_4 &= \frac{1}{12} u_0 (2u_0 + 3q_0 - 12p_0^2) \end{aligned} \quad (5a)$$

$$f_5 = -\frac{1}{4} u_0 p_0 (2u_0 + 3q_0 - 4p_0^2)$$

$$f_6 = -\left[\frac{1}{180} u_0^2 (2u_0 + 24q_0 - 186p_0^2) + \frac{1}{8} u_0 (q_0^2 + 8p_0^4 - 12p_0^2 q_0) \right]$$

$$\begin{aligned} g_3 &= -\frac{1}{3} u_0 \\ g_4 &= \frac{1}{2} u_0 p_0 \end{aligned} \quad (5b)$$

$$g_5 = \frac{1}{60} u_0 (2u_0 + 9q_0 - 36p_0^2)$$

$$g_6 = -\frac{1}{12} u_0 p_0 (2u_0 + 6q_0 - 8p_0^2)$$

We will now investigate the validity of the series expansion(4), following the general method used by Moulton [1], Charlier [2], and Andoyer [3], in their investigations of the radius of convergence of power series expansions for the Cartesian coordinates of planetary motion. In particular, Moulton gives a table of the radius of convergence as a function of the eccentricity e and the value of mean anomaly at time t_0 , M_0 , for elliptic orbits with a semimajor axis a of 2.65 astronomical units. This corresponds to the average orbit of the minor planets.

We have computed similar tables, corresponding to orbits of artificial Earth satellites and lunar probe trajectories. The following section outlines the analytical method employed in constructing these tables.

2. In lieu of the power-series expansions (4), consider the closed expressions for f and g given by Kühnert [4],

$$f = 1 - \frac{2\gamma^2}{r_0} \quad (6)$$

$$g = 2\gamma(r_0 \cos \varepsilon + \sigma_0 \gamma)$$

where

$$\varepsilon = \frac{1}{2}(E - E_0) \quad (7)$$

$$\gamma = a^{1/2} \sin \varepsilon$$

Since f and g are entire functions of ε , they may be written as power series in ε , which converge for all ε . Combining these series, we can write

$$\xi = \sum_{k=0}^{\infty} a_k \varepsilon^k \quad (8)$$

$$\eta = \sum_{k=0}^{\infty} b_k \varepsilon^k$$

Since ε is a function of E , and E a function of M , we can write

$$\varepsilon = \sum_{k=0}^{\infty} c_k (M - M_0)^k \quad (9)$$

Let R_{M_0} be the radius of convergence of this power series. By inserting (9) into (8) and rearranging terms, we can express ξ and η as power series of $M - M_0$, or equivalently, as power series of $\tau = t - t_0$. This would lead again to the Lagrangian expansions (4). Hence, for any M such that

$$|M - M_0| < R_{M_0} \quad (10)$$

the power series (4) are convergent.

Thus, the problem may be reduced to finding the radius of convergence R_{M_0} of series (9). This radius of convergence is the distance between M_0 and the singularity of the function $E(M)$ closest to M_0 . Since $E(M)$ is implicitly defined by

$$M = E - e \sin E \quad (11)$$

the singularities of $E(M)$ satisfy the equation

$$1 - e \cos E = 0 \quad (12)$$

It is easy to show that these singularities are given by

$$E_{\pm k} = 2k\pi \pm i\alpha \quad (13)$$

where k is any integer, and α is defined by

$$\cosh \alpha = \frac{1}{e} \quad (14)$$

We may therefore write

$$M_{\pm k} = 2k\pi \pm i(\alpha - \tanh \alpha) \quad (15)$$

By restricting M_0 to the interval $[-\pi, \pi]$, we obtain the following expression for the radius of convergence:

$$R_{M_0} = \sqrt{M_0^2 + (\alpha - \tanh \alpha)^2} \quad (16)$$

R_{M_0} may now be written as a function of time:

$$R_{M_0}^{(t)} = a^{3/2} R_{M_0} \quad (\text{in normalized units}) \quad (17)$$

It is worth noting that $\alpha - \tanh \alpha$ can be written as an explicit function of e ,

$$\alpha - \tanh \alpha = \ln \left(\frac{1 + \sqrt{1-e^2}}{e} \right) - \sqrt{1-e^2} \quad (18)$$

3. The tables of radii of convergence were constructed in three steps.

First, formula (18) was evaluated for different values of the eccentricity e ($0 \leq e < 1$). Then for each value of e , formula (16) was applied for different values of M_0 ($0 \leq M_0 \leq \pi$). Finally, for each pair of values (e, M_0) , the radius $R_{M_0}^{(t)}$ was computed by formula (17), for different values of a .

The ranges of e and a for which tables were constructed were chosen to correspond to typical orbits of artificial Earth Satellites and lunar probes, the latter being very elongated ellipses.

The two tables found in the Appendix are:

Table I. (Earth Satellite orbits):	$M_0 = 0^\circ (18^\circ) 180^\circ$
	$e = .05 (.05) .40$
	$a = 1.1 (.1) 2.0$
Table II. (Lunar trajectories):	$M_0 = 0^\circ (1^\circ) 10^\circ$
	$e = .85 (.01) .98$
	$a = 30. (1.) 40.$

These tables were computed on an IBM 7090 computer.

B. Truncation Error Investigation

The truncation error in the time-series expansions of the coordinates ξ and η can easily be estimated by evaluating the differences

$$|\xi - \xi^{(n)}| \quad \text{and} \quad |\eta - \eta^{(n)}|$$

where ξ, η are values obtained from the closed expressions (6), and $\xi^{(n)}, \eta^{(n)}$ are obtained from the time-series expansions (4), using only terms up to order ϵ^n , where n is specified. Practically, we have evaluated the truncation errors corresponding to the cubic and quintic approximations to the f and g series, for typical Earth Satellite orbits, with preassigned values of the semi-major axis a , and eccentricity e .

Similarly, the truncation error in the time-series expansions of the velocity components $\dot{\xi}$ and $\dot{\eta}$ may be estimated by evaluating the differences

$$|\dot{\xi} - \dot{\xi}^{(n)}| \quad \text{and} \quad |\dot{\eta} - \dot{\eta}^{(n)}|$$

where $\dot{\xi}$ and $\dot{\eta}$ are given by

$$\begin{aligned} \dot{\xi} &= \dot{f} \xi_0 + \dot{g} \dot{\xi}_0, & \dot{\xi} &= -\frac{2\dot{\eta} \cos E}{r_0 r} \\ \dot{\eta} &= \dot{f} \eta_0 + \dot{g} \dot{\eta}_0, & \dot{g} &= 1 - \frac{2\dot{\eta}^2}{r} \end{aligned}$$

and $\dot{\xi}^{(n)}$ and $\dot{\eta}^{(n)}$ are the derivatives of the truncated series of order n .

To start the computations, it has been assumed that the satellite is at perigee (i. e. $M_0 = 0$), at origin of time t_0 . At perigee, the initial values of position and velocity are

$$\begin{aligned} \xi_0 &= a(1-e), & \dot{\xi} &= 0 \\ \eta_0 &= 0, & \dot{\eta} &= \sqrt{\frac{1+e}{\xi_0}} \end{aligned}$$

The closed expressions and truncated power series are then evaluated at equally spaced intervals of time t_i ,

$$t_i = t_0 + i \Delta t \quad (\Delta t = 5 \text{ sec.}, i = 0 \rightarrow 60)$$

The same general computation is then repeated taking another value of M_0 as the origin, and recomputing the initial position and velocity.

The results of these computations have been plotted in Figures 1 - 5 of the appendix. Graphs of $|r - r^{(n)}|$, $|V - V^{(n)}|$, and $d\psi^{(n)}$ are given, where

$$\begin{aligned} r &= (\xi^2 + \eta^2)^{1/2}, & r^{(n)} &= (\xi^{(n)2} + \eta^{(n)2})^{1/2} \\ V &= (\dot{\xi}^2 + \dot{\eta}^2)^{1/2}, & V^{(n)} &= (\dot{\xi}^{(n)2} + \dot{\eta}^{(n)2})^{1/2} \end{aligned}$$

and $d\psi^{(n)}$ is the angle between the position vectors (ξ, η) and $(\xi^{(n)}, \eta^{(n)})$.

Since the truncation error is largest for $M_0 = 0$, where the radius of convergence of the power series is the smallest, the graphs of the truncation error as a function of time have been provided for this case.

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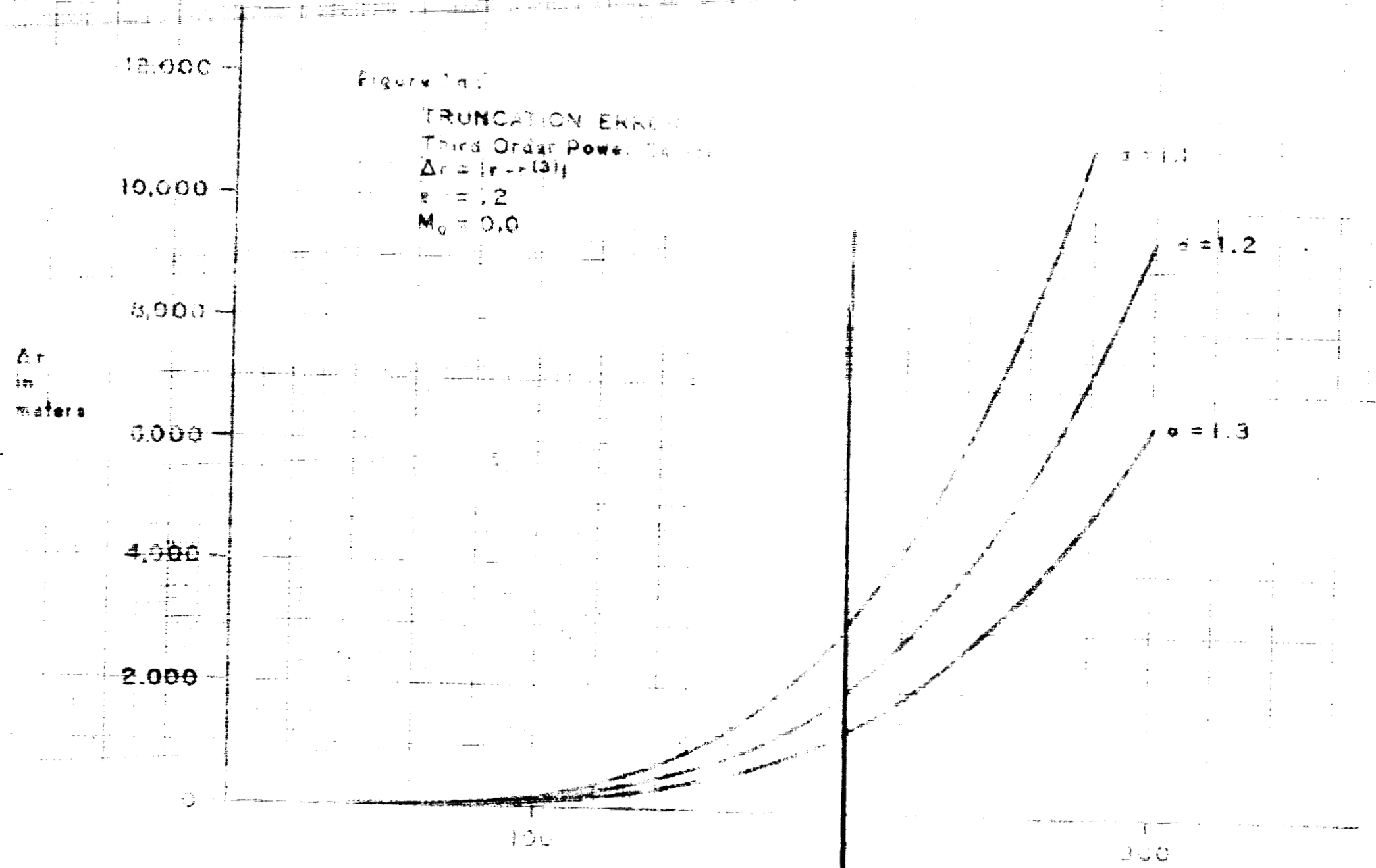
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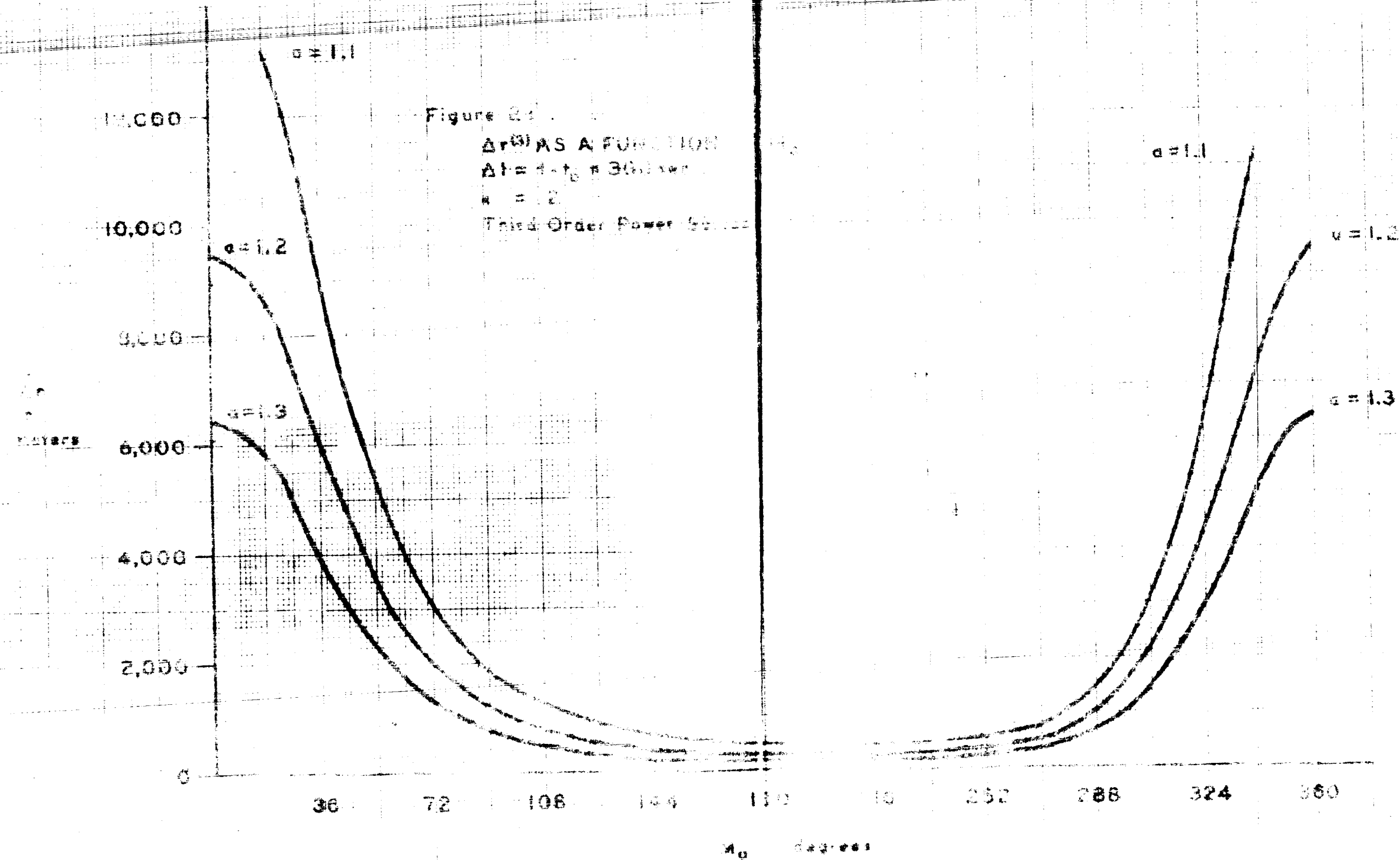
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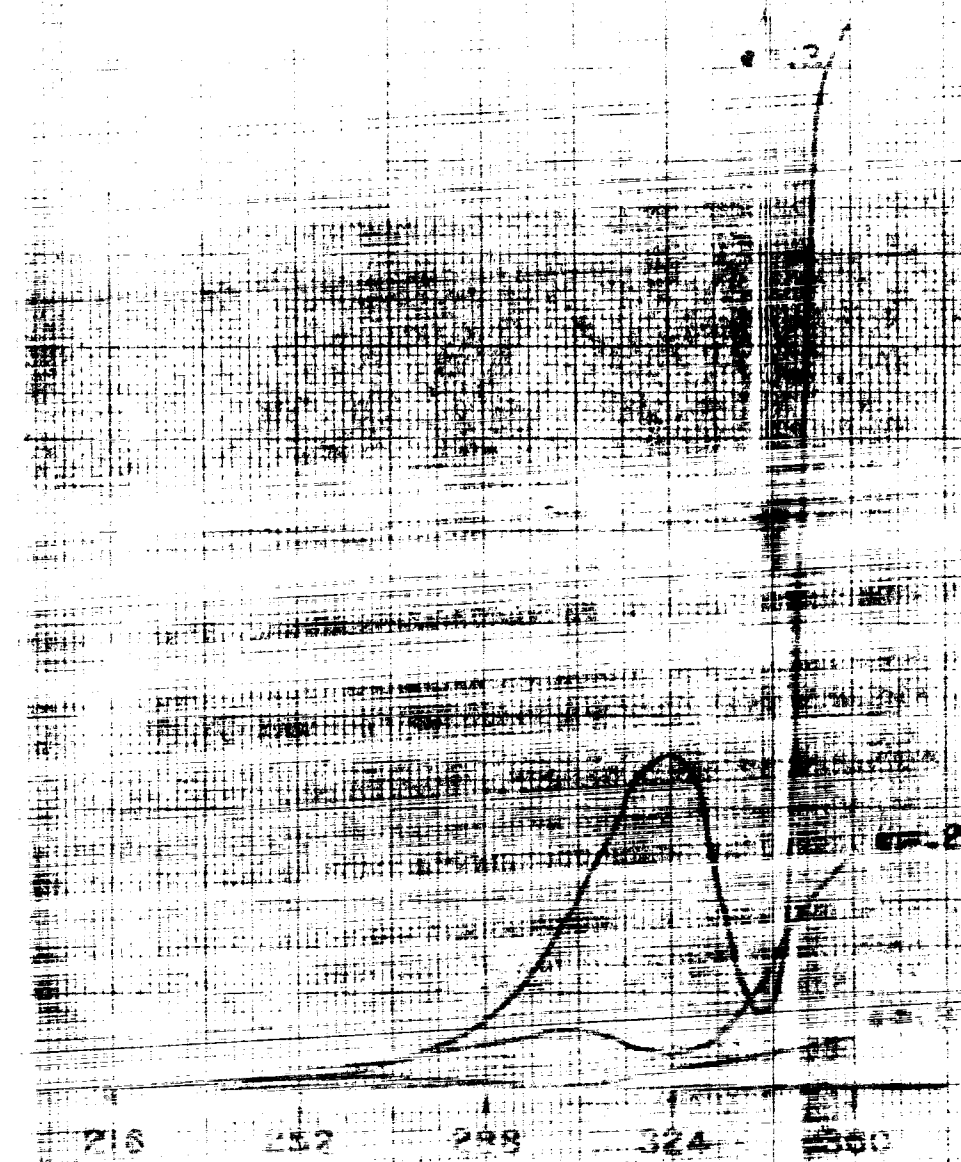
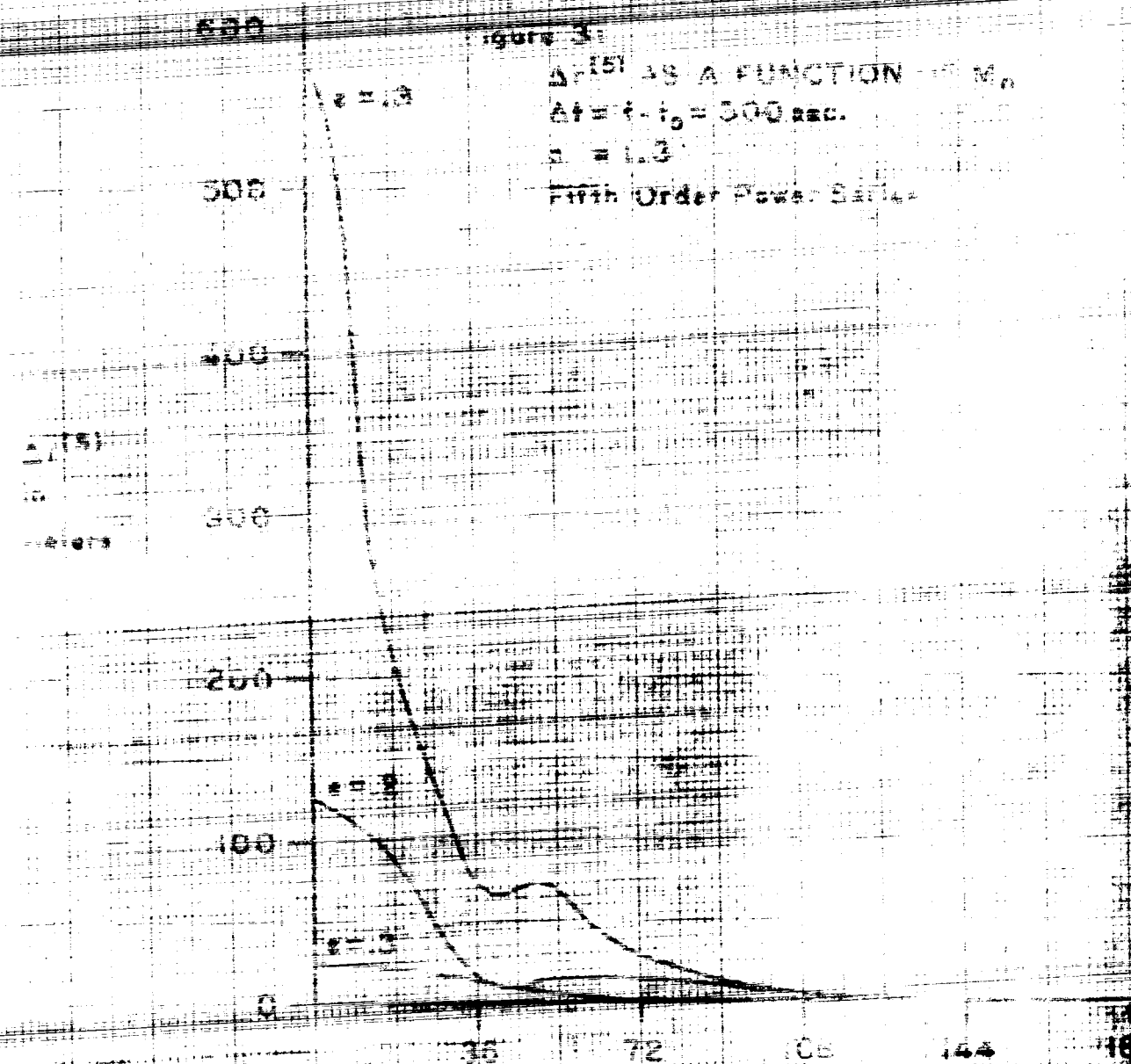
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